Indian Statistical Institute, Bangalore

B. Math. First Year, Second Semester

Probability Theory-II Mid-term Examination

Duration: 3 hours Maximum marks: 100 Date : 02-03-2012

1. We have 100 balls numbered 1 to 100, and two urns named A, B. Each ball is put in one of the urns at random. (i) Suppose you opt to take all the balls in urn B, what is the expected number of balls you get? (ii) Suppose you opt to take all the balls in the urn which got the ball numbered 77, what is the expected number of balls you get? [10]

2. A string of unit length is cut into two pieces by cutting at a point uniformly at random. Let M be the length of the piece containing the midpoint of the string. Find the distribution and density, and expectation of M. [15]

3. Fix $0 < p < 1$. A random variable X is said to have geometric distribution with parameter p, if X takes values in $\{1, 2, \ldots\}$ and $P(X = k) = pq^{k-1}$ for $k = 1, 2, \ldots$. Suppose X, Y are independent geometric random variables with parameter p, show that $\mathrm{Min}(X, Y)$ and $X - Y$ are independent. [15]

4. Let (S, T) be two random variables with joint density given by:

$$f(s,t) = \begin{cases} k & \text{for } \ 0 \leq s \leq a, 0 \leq t \leq b \\ 0 & \text{otherwise} \end{cases},$$

where a, b, k are positive constants.

(i) Determine the constant k (in terms of a, b.).

(ii) Write down the joint distribution function of (S, T).

(iii) Find marginal densities of S, T. Are S, T independent?

(iv) Find conditional density of S given $T = t$ for $0 \leq t \leq b$. [20]

P.T.O.

5. Let X_1, X_2 be i.i.d. random variables with common density:

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} \qquad -\infty < x < \infty.$$

Take $Y_1 = 4X_1 + 6X_2$ and $Y_2 = -3X_1 + 2X_2$. Find the joint density of Y_1, Y_2. Find the marginal density of Y_1.

[20]

6. Fix $\lambda > 0$. A random variable M is said to have exponential distribution with parameter λ, if it has density g given by $g(t) = \lambda e^{-\lambda t}\chi_{(0,\infty)}(t)$. Suppose M, N are independent exponential random variables with parameter λ, find the density of $\frac{M}{M+N}$. [20]

7. Suppose R is a random variable with probability density given by

$$g(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Compute the moment generating and characteristic functions of R. [10]